Integrable systems in projective differential geometry

Ferapontov E.V.
Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom
and
Centre for Nonlinear Studies
Landau Institute of Theoretical Physics
Academy of Science of Russia, Kosygina 2
117940 Moscow, GSP-1, Russia
e-mail: `fer@landau.ac.ru`

Abstract

Some of the most important classes of surfaces in projective 3-space are reviewed: these are isothermally asymptotic surfaces, projectively applicable surfaces, surfaces of Jonas, projectively minimal surfaces, etc. It is demonstrated that the corresponding projective "Gauss-Codazzi" equations reduce to integrable systems which are quite familiar from the modern soliton theory and coincide with the stationary flows in the Davey-Stewartson and Kadomtsev-Petviashvili hierarchies, equations of the Toda lattice, etc. The corresponding Lax pairs can be obtained by inserting a spectral parameter in the equations of the Wilczynski moving frame.

Keywords: projective differential geometry, integrable systems.

Mathematical Subject Classification: 53A40, 53A20, 35Q58, 35L60.

1 Introduction

Projective differential geometry of surfaces M^2 in P^3 has been extensively developed in the first half of the century in the works of Wilczynski, Fubini, Čech, Cartan, Tzitzeica, Demoulin, Rozet, Godeaux, Lane, Eisenhart, Finikov, Bol and many others. These investigations culminated in a number of beautifull geometric constructions and a list of particularly interesting classes of surfaces which now seem to be quite deeply forgotten. Our aim here is to give a brief review of surfaces in projective 3-space with the emphasize on the nonlinear equations underlying them. In particular, we demonstrate that

isothermally asymptotic surfaces are described by the stationary modified Veselov-Novikov (mVN) equation (sect.3);

projectively applicable surfaces (which naturally fall into the classes of the so-called "surfaces R_0" and "surfaces R") correspond to the stationary limits of the second flow in the Kadomtsev-Petviashvili (KP) hierarchy and the stationary Davey-Stewartson (DS) system, respectively (sect.4);

surfaces of Jonas are related to a stationary limit of the fourth order flow in the DS hierarchy (sect.5);

projectively minimal surfaces are governed by an integrable system which, in a certain limit, reduces to the coupled Tzitzeica system, being a reduction of periodic Toda lattice of period 6 (sect.6);

Surfaces with asymptotic lines in linear complexes correspond to a linearizable system (sect.7).

Thus, all of the most important classes of surfaces in projective differential geometry are described by integrable systems which are an object of the modern soliton theory. This is by no means surprising: the reason for differential geometers to introduce a particular class of surfaces has always been the existence of certain "nice geometric transformations" allowing the construction of new surfaces from the given ones. On the other hand, the existence of Bäcklund transformations is a standard hint on the integrability. Thus, in a sense, classical differential geometry (here we mean geometry of surfaces in 3-space) can be viewed as a chapter of the theory of integrable systems. Most of the Bäcklund transformations derived in the "solitonic" context can be found in the old geometric textbooks (probably, stated in geometric terms, without an explicit coordinate formulae). In sect.7 we review the construction of congruences W, which are the main source of Bäcklund transformations. It requires a solution of certain Dirac equation on the surface M^2. Since Dirac operator plays a role of the Lax operator of the (2+1)-dimensional DS hierarchy, this construction clarifies the relationship between surfaces in projective differential geometry and stationary flows of the DS hierarchy.

In our approach we make use of the Wilczynski moving frame which proves to be the most convenient tool for studying surfaces in projective 3-space. For instance, the Lax pairs for systems governing particular classes of surfaces can be obtained by inserting an appropriate spectral parameter in the equations of motion of the Wilczynski frame.

2 Surfaces in projective differential geometry

Based on [59] (see also [12], [46], [47]), let us briefly recall the standard way of defining surfaces M^2 in projective space P^3 in terms of solutions of a linear system

$$\begin{aligned} \mathbf{r}_{xx} &= \beta\ \mathbf{r}_y + \tfrac{1}{2}(V - \beta_y)\ \mathbf{r} \\ \mathbf{r}_{yy} &= \gamma\ \mathbf{r}_x + \tfrac{1}{2}(W - \gamma_x)\ \mathbf{r} \end{aligned} \tag{1}$$

where β, γ, V, W are functions of x and y. If we cross-differentiate (1) and assume $\mathbf{r}, \mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_{xy}$ to be independent, we arrive at the compatibility conditions [33, p. 120]

$$\begin{aligned} \beta_{yyy} - 2\beta_y W - \beta W_y &= \gamma_{xxx} - 2\gamma_x V - \gamma V_x \\ W_x &= 2\gamma\beta_y + \beta\gamma_y \\ V_y &= 2\beta\gamma_x + \gamma\beta_x. \end{aligned} \tag{2}$$

For any fixed β, γ, V, W satisfying (2) the linear system (1) is compatible and possesses a solution $\mathbf{r} = (r^0, r^1, r^2, r^3)$ where $r^i(x, y)$ can be regarded as homogeneous coordinates of a surface in projective space P^3. One may think of M^2 as a surface in a three-dimensional space with position vector $\mathbf{R} = (r^1/r^0, r^2/r^0, r^3/r^0)$. If we choose any other solution $\tilde{\mathbf{r}} = (\tilde{r}^0, \tilde{r}^1, \tilde{r}^2, \tilde{r}^3)$ of the same system (1) then the corresponding surface $\tilde{M}^2$ with position vector $\tilde{\mathbf{R}} = (\tilde{r}^1/\tilde{r}^0, \tilde{r}^2/\tilde{r}^0, \tilde{r}^3/\tilde{r}^0)$ constitutes a projective transform of M^2 so that any fixed β, γ, V, W satisfying (2) define a surface M^2 uniquely up to projective equivalence. Moreover, a simple calculation yields

$$\begin{aligned} \mathbf{R}_{xx} &= \beta\ \mathbf{R}_y + a\ \mathbf{R}_x \\ \mathbf{R}_{yy} &= \gamma\ \mathbf{R}_x + b\ \mathbf{R}_y \end{aligned} \tag{3}$$

($a = -2r^0_x/r^0$, $b = -2r^0_y/r^0$) which implies that x, y are asymptotic coordinates of the surface M^2. System (3) can be viewed as an "affine gauge" of system (1). In what follows, we assume that our surfaces are hyperbolic and the corresponding asymptotic coordinates x, y are real.[1] Since equations (2) specify a surface uniquely up to projective equivalence, they can be viewed as the 'Gauss-Codazzi' equations in projective geometry.

Even though the coefficients β, γ, V, W define a surface M^2 uniquely up to projective equivalence via (1), it is not entirely correct to regard β, γ, V, W as projective invariants. Indeed, the asymptotic coordinates x, y are only defined up to an arbitrary reparametrization of the form

$$x^* = f(x), \qquad y^* = g(y) \tag{4}$$

which induces a scaling of the surface vector according to

$$\mathbf{r}^* = \sqrt{f'(x)g'(y)}\ \mathbf{r}. \tag{5}$$

Thus [6, p. 1], the form of equations (1) is preserved by the above transformation with the new coefficients $\beta^*, \gamma^*, V^*, W^*$ given by

$$\begin{aligned} \beta^* &= \beta g'/(f')^2, \qquad V^*(f')^2 = V + S(f) \\ \gamma^* &= \gamma f'/(g')^2, \qquad W^*(g')^2 = W + S(g), \end{aligned} \tag{6}$$

[1] The elliptic case is dealt with in an analogous manner by regarding x, y as complex conjugates.

where $S(\cdot)$ is the Schwarzian derivative, that is

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The transformation formulae (6) imply that the symmetric 2-form

$$2\beta\gamma\, dxdy$$

and the conformal class of the cubic form

$$\beta\, dx^3 + \gamma\, dy^3$$

are absolute projective invariants. They are known as the projective metric and the Darboux cubic form, respectively, and play an important role in projective differential geometry. In particular, they define a 'generic' surface uniquely up to projective equivalence. The vanishing of the Darboux cubic form is characteristic for quadrics: indeed, in this case $\beta = \gamma = 0$ so that asymptotic curves of both families are straight lines. The vanishing of the projective metric (which is equivalent to either $\beta = 0$ or $\gamma = 0$) characterises ruled surfaces. In what follows we exclude these two degenerate situations and require $\beta \neq 0$, $\gamma \neq 0$.

One can also define projectively invariant differentials

$$\omega^1 = (\gamma\beta^2)^{\frac{1}{3}}\, dx, \quad \omega^2 = (\beta\gamma^2)^{\frac{1}{3}}\, dy,$$

so that $\beta\gamma\, dxdy = \omega^1\omega^2$. With the help of ω^1, ω^2 one can define projectively invariant differentiation. However, we will not take advantage of it in what follows.

Using (4)-(6), one can verify that the four points

$$\begin{gathered} \mathbf{r}, \quad \mathbf{r}_1 = \mathbf{r}_x - \tfrac{1}{2}\tfrac{\gamma_x}{\gamma}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{r}_y - \tfrac{1}{2}\tfrac{\beta_y}{\beta}\mathbf{r}, \\ \boldsymbol{\eta} = \mathbf{r}_{xy} - \tfrac{1}{2}\tfrac{\gamma_x}{\gamma}\mathbf{r}_y - \tfrac{1}{2}\tfrac{\beta_y}{\beta}\mathbf{r}_x + \left(\tfrac{1}{4}\tfrac{\beta_y\gamma_x}{\beta\gamma} - \tfrac{1}{2}\beta\gamma\right)\mathbf{r} \end{gathered} \tag{7}$$

are defined in an invariant way, that is under the transformation formulae (4)-(6) they acquire a nonzero multiple which does not change them as points in projective space P^3. These points form the vertices of the so-called Wilczynski moving tetrahedral [6], [14], [59]. Since the lines passing through $\mathbf{r}, \mathbf{r}_1$ and $\mathbf{r}, \mathbf{r}_2$ are tangential to the x- and y-asymptotic curves, respectively, the three points $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$ span the tangent plane of the surface M^2. The line through $\mathbf{r}_1, \mathbf{r}_2$ lying in the tangent plane is known as the directrix of Wilczynski of the second kind. The line through $\mathbf{r}, \boldsymbol{\eta}$ is transversal to M^2 and is known as the directrix of Wilczynski of the first kind. It plays the role of a projective 'normal'. We stress that in projective differential geometry there exists no unique choice of an invariant normal. This is in contrast with Euclidean and affine geometries in which the normal is canonically defined. Some of the best-known and most-investigated normals are those of Wilczynski, Fubini, Green, Darboux, Bompiani and Sullivan [6, p. 35] with the directrix of Wilczynski being the most commonly used. It is known that the normal of Wilczynski intersects the tangent Lie quadric of the surface M^2 at exactly two points $\mathbf{r}$ and $\boldsymbol{\eta}$ so that both points

lie on the Lie quadric and are canonically defined. The Wilczynski tetrahedral proves to be the most convenient tool in projective differential geometry.

Using (1) and (7), we easily derive for $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\eta}$ the linear equations [14, p. 42]

$$\begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}\frac{\gamma_x}{\gamma} & 1 & 0 & 0 \\ \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma} & \beta & 0 \\ \frac{1}{2}k & 0 & \frac{1}{2}\frac{\gamma_x}{\gamma} & 1 \\ \frac{1}{2}\beta a & \frac{1}{2}k & \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}_y = \begin{pmatrix} \frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1 & 0 \\ \frac{1}{2}l & \frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1 \\ \frac{1}{2}a & \gamma & -\frac{1}{2}\frac{\beta_y}{\beta} & 0 \\ \frac{1}{2}\gamma b & \frac{1}{2}a & \frac{1}{2}l & -\frac{1}{2}\frac{\beta_y}{\beta} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}, \tag{8}$$

where we introduced the notation

$$k = \beta\gamma - (\ln\beta)_{xy}, \qquad l = \beta\gamma - (\ln\gamma)_{xy},$$
$$a = W - (\ln\beta)_{yy} - \tfrac{1}{2}(\ln\beta)_y^2, \qquad b = V - (\ln\gamma)_{xx} - \tfrac{1}{2}(\ln\gamma)_x^2. \tag{9}$$

Under the transformations (4)-(6) these quantities transform as follows

$$k^* = k/f'g', \qquad l^* = l/f'g',$$
$$a^* = a/(g')^2, \qquad b^* = b/(f')^2, \tag{10}$$

and give rise to the projectively invariant quadratic form

$$b\,dx^2 + a\,dy^2$$

and the quartic form

$$a\beta^2 dx^4 + b\gamma^2 dy^4.$$

The compatibility conditions of equations (8) imply

$$(\ln\beta)_{xy} = \beta\gamma - k, \qquad (\ln\gamma)_{xy} = \beta\gamma - l,$$
$$a_x = k_y + \tfrac{\beta_y}{\beta}k, \qquad b_y = l_x + \tfrac{\gamma_x}{\gamma}l, \tag{11}$$
$$\beta a_y + 2a\beta_y = \gamma b_x + 2b\gamma_x,$$

which is just the equivalent form of the projective "Gauss-Codazzi" equations (2).

Equations (8) can be rewritten in the Plücker coordinates. For a convenience of the reader we briefly recall this construction. Let us consider a line l in P^3 passing through the points $\mathbf{a}$ and $\mathbf{b}$ with the homogeneous coordinates $\mathbf{a} = (a^0 : a^1 : a^2 : a^3)$ and $\mathbf{b} = (b^0 : b^1 : b^2 : b^3)$. With the line l we associate a point $\mathbf{a} \wedge \mathbf{b}$ in projective space P^5 with the homogeneous coordinates

$$\mathbf{a} \wedge \mathbf{b} = (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}),$$

where

$$p_{ij} = \det \begin{pmatrix} a^i & a^j \\ b^i & b^j \end{pmatrix}.$$

The coordinates p_{ij} satisfy the well-known quadratic Plücker relation

$$p_{01}\, p_{23} + p_{02}\, p_{31} + p_{03}\, p_{12} = 0. \tag{12}$$

Instead of $\mathbf{a}$ and $\mathbf{b}$ we may consider an arbitrary linear combinations thereof without changing $\mathbf{a} \wedge \mathbf{b}$ as a point in P^5. Hence, we arrive at the well-defined Plücker coorrespondence $l(\mathbf{a}, \mathbf{b}) \to \mathbf{a} \wedge \mathbf{b}$ between lines in P^3 and points on the Plücker quadric in P^5. Plücker correspondence plays an important role in the projective differential geometry of surfaces and often sheds some new light on those properties of surfaces which are not 'visible' in P^3 but acquire a precise geometric meaning only in P^5. Thus, let us consider a surface $M^2 \in P^3$ with the Wilczynski tetrahedral $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\eta}$ satisfying equations (8). Since the two pairs of points $\mathbf{r}, \mathbf{r}_1$ and $\mathbf{r}, \mathbf{r}_2$ generate two lines in P^3 which are tangential to the x- and y-asymptotic curves, respectively, the formulae

$$\mathcal{U} = \mathbf{r} \wedge \mathbf{r}_1, \quad \mathcal{V} = \mathbf{r} \wedge \mathbf{r}_2$$

define the images of these lines under the Plücker embedding. Hence, with any surface $M^2 \in P^3$ there are canonically associated two surfaces $\mathcal{U}(x, y)$ and $\mathcal{V}(x, y)$ in P^5 lying on the Plücker quadric (12). In view of the formulae

$$\mathcal{U}_x = \beta\, \mathcal{V}, \quad \mathcal{V}_y = \gamma\, \mathcal{U},$$

we conclude that the line in P^5 passing through a pair of points $(\mathcal{U}, \mathcal{V})$ can also be generated by the pair of points $(\mathcal{U}, \mathcal{U}_x)$ (and hence is tangential to the x-coordinate line on the surface $\mathcal{U}$) or by a pair of points $(\mathcal{V}, \mathcal{V}_y)$ (and hence is tangential to the y-coordinate line on the surface $\mathcal{V}$). Consequently, the surfaces $\mathcal{U}$ and $\mathcal{V}$ are two focal surfaces of the congruence of straight lines $(\mathcal{U}, \mathcal{V})$ or, equivalently, $\mathcal{V}$ is the Laplace transform of $\mathcal{U}$ with respect to x and $\mathcal{U}$ is the Laplace transform of $\mathcal{V}$ with respect to y. We emphasize that the x- and y-coordinate lines on the surfaces $\mathcal{U}$ and $\mathcal{V}$ are not asymptotic but conjugate. Continuation of the Laplace sequence in both directions, that is taking the x-transform of $\mathcal{V}$, the y-transform of $\mathcal{U}$, etc., leads, in the generic case, to an infinite Laplace sequence in P^5 known as the Godeaux sequence of a surface M^2 [6, p. 344]. The surfaces of the Godeaux sequence carry important geometric information about the surface M^2 itself.

The case of a closed, i.e. periodic Godeaux sequence is particularly interesting. It turns out, that the only surfaces $M^2 \in P^3$ for which the Godeaux sequence is of period 6 (the value 6 turns out to be the least possible) are the surfaces of Demoulin [6, p. 360] – see sect.4.

Introducing

$$\mathcal{A} = \mathbf{r}_2 \wedge \mathbf{r}_1 + \mathbf{r} \wedge \boldsymbol{\eta}, \quad \mathcal{B} = \mathbf{r}_1 \wedge \mathbf{r}_2 + \mathbf{r} \wedge \boldsymbol{\eta},$$

$$\mathcal{P} = 2\, \mathbf{r}_2 \wedge \boldsymbol{\eta}, \quad \mathcal{Q} = 2\, \mathbf{r}_1 \wedge \boldsymbol{\eta},$$

we arrive at the following equations for the Plücker coordinates:

$$
\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_x}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ -\beta a & 0 & \beta & 0 & b & -\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}
$$

$$
\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\gamma b & 0 & \gamma \\ \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix} \tag{13}
$$

Equations (13) are consistent with the following table of scalar products:

$$
(\mathcal{U}, \mathcal{P}) = -1, \quad (\mathcal{A}, \mathcal{A}) = 1, \quad (\mathcal{V}, \mathcal{Q}) = 1, \quad (\mathcal{B}, \mathcal{B}) = -1, \tag{14}
$$

all other scalar products being equal to zero. This defines a scalar product of the signature (3, 3) which is the same as that of the quadratic form (12).

Different types of surfaces can be defined by imposing additional constraints on β, γ, V, W (respectively, $\beta, \gamma, k, l, a, b$), so that, in a sense, projective differential geometry is the theory of (integrable) reductions of the underdetermined system (2) (respectively, (11)). Although the three linear systems (1), (8) and (13) are in fact equivalent, some of them prove to be more suitable for studying particular classes of projective surfaces.

Remark. Since the tangent plane of the surface M^2 is spanned by three points $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$, the vector $\mathbf{r}^d = \mathbf{r} \wedge \mathbf{r}_1 \wedge \mathbf{r}_2$ can be viewed as a radius-vector of the dual surface. A simple calculation yields

$$
\mathbf{r}^d_{xx} = -\beta \, \mathbf{r}^d_y + \tfrac{1}{2}(V + \beta_y) \, \mathbf{r}^d
$$

$$
\mathbf{r}^d_{yy} = -\gamma \, \mathbf{r}^d_x + \tfrac{1}{2}(W + \gamma_x) \, \mathbf{r}^d
$$

implying that the passage to the dual surface is equivalent to a simple change of signs: $\beta, \gamma, V, W \ \rightarrow \ -\beta, -\gamma, V, W$. This transformation is obviously a discrete symmetry of equations (2).

3 Isothermally asymptotic surfaces

These surfaces are specified by the condition

$$
\left(\ln \frac{\beta}{\gamma} \right)_{xy} = 0,
$$

which, in view of the transformation formulae (6), reduces to $\beta = \gamma$ after a suitable choice of coordinates x, y. In this case equations (2) assume the form of the stationary modified

Veselov-Novikov (mVN) equation

$$\beta_{yyy}-2\beta_y W-\beta W_y=\beta_{xxx}-2\beta_x V-\beta V_x,$$
$$W_x=\tfrac{3}{2}(\beta^2)_y \tag{15}$$
$$V_y=\tfrac{3}{2}(\beta^2)_x.$$

In this case one can introduce a spectral parameter λ in equations (13) without violating their compatibility:

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}_x=\begin{pmatrix}0&0&0&\beta&0&0\\ k&0&0&0&0&0\\ 0&k&0&-\beta a&0&0\\ 0&0&0&\frac{\beta_x}{\beta}&1&0\\ 0&0&0&b&0&1\\ -\beta a&0&\beta&\lambda&b&-\frac{\beta_x}{\beta}\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}$$
$$\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}_y=\begin{pmatrix}\frac{\beta_y}{\beta}&1&0&0&0&0\\ a&0&1&0&0&0\\ -\lambda&a&-\frac{\beta_y}{\beta}&-\beta b&0&\beta\\ \beta&0&0&0&0&0\\ 0&0&0&l&0&0\\ -\beta b&0&0&0&l&0\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix} \tag{16}$$

Rewriting (16) in terms of $\mathcal{U},\mathcal{V}$ (that is, expressing $\mathcal{A},\mathcal{B},\mathcal{P},\mathcal{Q}$ through $\mathcal{U}$,$\mathcal{U}_y$,$\mathcal{U}_{yy}$ and $\mathcal{V},\mathcal{V}_x,\mathcal{V}_{xx}$), we arrive at

$$\begin{gathered}\mathcal{U}_x=\beta\ \mathcal{V}\\ \mathcal{V}_y=\beta\ \mathcal{U}\\ \lambda\mathcal{U}=\mathcal{U}_{xxx}-\mathcal{U}_{yyy}+2\ W\ \mathcal{U}_y-3\ \beta_x\ \mathcal{V}_x+W_y\ \mathcal{U}-2\ \beta\ V\ \mathcal{V}\\ \lambda\mathcal{V}=\mathcal{V}_{xxx}-\mathcal{V}_{yyy}-2\ V\ \mathcal{V}_x+3\ \beta_y\ \mathcal{U}_y-V_x\ \mathcal{V}+2\ \beta\ W\ \mathcal{U}.\end{gathered} \tag{17}$$

which coincides with the stationary limit of the mVN linear problem (the compatibility conditions of (17) coincide with (15)).

We recall that the (2+1)-dimensional mVN equation

$$\beta_t=\beta_{xxx}-\beta_{yyy}-2\ \beta_x\ V+2\ \beta_y\ W-\beta\ V_x+\beta\ W_y$$
$$W_x=\tfrac{3}{2}(\beta^2)_y \tag{18}$$
$$V_y=\tfrac{3}{2}(\beta^2)_x$$

was introduced in [5] and is associated with the two-dimensional Dirac operator

$$\begin{aligned} \mathcal{U}_x &= \beta\, \mathcal{V} \\ \mathcal{V}_y &= \beta\, \mathcal{U} \end{aligned}$$

$$\begin{aligned} \mathcal{U}_t &= \mathcal{U}_{xxx} - \mathcal{U}_{yyy} + 2\ W\ \mathcal{U}_y - 3\ \beta_x\ \mathcal{V}_x + W_y\ \mathcal{U} - 2\ \beta\ V\ \mathcal{V} \\ \mathcal{V}_t &= \mathcal{V}_{xxx} - \mathcal{V}_{yyy} - 2\ V\ \mathcal{V}_x + 3\ \beta_y\ \mathcal{U}_y - V_x\ \mathcal{V} + 2\ \beta\ W\ \mathcal{U}. \end{aligned} \tag{19}$$

Linear system (17) results after a substitution $\mathcal{U}_t \to \lambda \mathcal{U}, \quad \mathcal{V}_t \to \lambda \mathcal{V}$ which is a standard way to introduce a spectral parameter in the stationary problem.

Isothermally asymptotic surfaces are attributed to Fubini [17] and can be equivalently defined by any of the following geometric properties:

- The 3-web, formed by asymptotic curves and Darboux's curves, is hexagonal. (Darboux's curves are the zero curves of the Darboux cubic form $\beta\, dx^3 + \gamma\, dy^3$).

- Isothermally asymptotic surfaces are the focal surfaces of special W-congruences, preserving Darboux's curves (in fact, this property implies a Bäcklund transformation for isothermally asymptotic surfaces – see sect.7).

Examples of isothermally asymptotic surfaces include arbitrary quadrics and cubics, quartics of Kummer, projective transforms of affine spheres and rotation surfaces:

Quadrics correspond to the trivial solution $\beta = 0,\ W = W(y),\ V = V(x)$.

Projective transforms of rotation surfaces are specified by $\beta = \beta(x+y),\ W = V = \frac{3}{2}\beta^2 + c$ where β is an arbitrary function of $(x+y)$ and c is an arbitrary constant. For $c > 0$ these are indeed projective transforms of surfaces $z = f(x^2+y^2)$, while the cases $c = 0$ and $c < 0$ correspond to projective transforms of surfaces $z = f(x^2+y)$ and $z = f(x^2-y^2)$, respectively. Travelling-wave solutions $\beta(x+cy)$ of equation (15) correspond to surfaces, which are invariant under one-parameter groups of projective transformations. In the case $c \neq 1$ the function β is no longer arbitrary and can be expressed in elliptic functions (compare with [36]).

Cubic surfaces are specified by the following additional constraints in (15):

$$V = -\frac{1}{2}(\ln\beta)_{xx} + \frac{1}{8}(\ln\beta)_x^2 + \frac{5}{2}\beta_y, \quad W = -\frac{1}{2}(\ln\beta)_{yy} + \frac{1}{8}(\ln\beta)_y^2 + \frac{5}{2}\beta_x \tag{20}$$

[34], see also [33], p.131. With these V, W equations (15) imply

$$\left(\frac{(\ln\beta)_{xy}}{\sqrt{\beta}} + 4\beta\sqrt{\beta}\right)_y = 5\frac{\beta_{xx}}{\sqrt{\beta}}, \quad \left(\frac{(\ln\beta)_{xy}}{\sqrt{\beta}} + 4\beta\sqrt{\beta}\right)_x = 5\frac{\beta_{yy}}{\sqrt{\beta}}.$$

Integration of these equations for β would provide a 4-parameter family of exact solutions of equation (15): indeed, up to projective equivalence cubics in P^3 depend on 4 essential parameters.

The Roman surface of Steiner is a rational quartic in P^3 with the equation

$$(x^2+y^2+z^2-1)^2 = ((z-1)^2 - 2x^2)((z+1)^2 - 2y^2)$$

owing it's name to Steiner who investigated this surface in Rome in 1844. Besides quadrics and ruled cubic surfaces the Roman surface of Steiner is the only surface in P^3 possessing infinitely many conic sections through any of it's points. This result was announced several times: by Moutard in 1865, Darboux in 1880 and Wilczynski in 1908 (see [59], 1909 for historical remarks). The interest to the Roman surface of Steiner in projective differential geometry is due to the remarkable construction of Darboux, relating with an arbitrary surface M^2 in P^3 and an arbitrary point p on M^2 an osculating Roman surface of Steiner which has the fourth order of tangency with M^2 at this point. Analytically, the Roman surface of Steiner corresponds to the choice

$$V=-\frac{1}{2}(\ln\beta)_{xx}+\frac{1}{8}(\ln\beta)_x^2-\frac{5}{2}\beta_y, \quad W=-\frac{1}{2}(\ln\beta)_{yy}+\frac{1}{8}(\ln\beta)_y^2-\frac{5}{2}\beta_x, \tag{21}$$

$$(\ln\beta)_{xy}=\frac{4}{9}\beta^2$$

([6], p.149-150) implying upon substitution in (15) the following equations for β:

$$\begin{aligned}\beta_{xx}&=-\tfrac{4}{3}\beta\beta_y\\ \beta_{yy}&=-\tfrac{4}{3}\beta\beta_x\\ (\ln\beta)_{xy}&=\tfrac{4}{9}\beta^2.\end{aligned}\tag{22}$$

These can be explicitely integrated:

$$\beta^2=\frac{9}{4}\frac{f'g'}{(f+g)^2}$$

where the functions $f(x)$ and $g(y)$ satisfy the ODE's

$$(f')^3=(a_0+a_1f+a_2f^2)^2, \quad (g')^3=(a_0-a_1g+a_2g^2)^2.$$

Here a_i are arbitrary constants. Under the transformation $(\beta,V,W)\to(-\beta,V,W)$ equations (21) transform to (20). This means, that the dual of the Roman surface of Steiner is a cubic, and hence the Roman surface itself is a quartic of class 3 ([6], p.150).

The Roman surface of Steiner belongs to a broader class of isothermally asymptotic quartic surfaces known as

Quartics of Kummer investigated by Kummer as singular surfaces of quadratic line complexes. Analytically, the quartics of Kummer are specified by the conditions

$$\begin{aligned}V=\tfrac{11}{8}(\ln\beta)_{xx}+2(\ln\beta)_x^2, \quad W=\tfrac{11}{8}(\ln\beta)_{yy}+2(\ln\beta)_y^2,\\ (\ln\beta)_{xy}=\tfrac{4}{9}\beta^2\end{aligned}\tag{23}$$

([6], p.231). Substituting these V,W in (15) we arrive at

$$\left(\frac{1}{\beta^2}(\beta^2(\beta^2)_y)_y\right)_y=\left(\frac{1}{\beta^2}(\beta^2(\beta^2)_x)_x\right)_x. \tag{24}$$

With

$$\beta^2=\frac{9}{4}\frac{f^{'}g^{'}}{(f+g)^2}$$

(which is a general solution of the Liouville equation $(23)_3$) equations (24) can be rewritten in the form

$$\begin{aligned}\tfrac{1}{3}(f+g)^3\left(\tfrac{d^3((f^{'})^3)}{df^3}-\tfrac{d^3((g^{'})^3)}{dg^3}\right)-4(f+g)^2\left(\tfrac{d^2((f^{'})^3)}{df^2}-\tfrac{d^2((g^{'})^3)}{dg^2}\right)+\\ 20(f+g)\left(\tfrac{d((f^{'})^3)}{df}-\tfrac{d((g^{'})^3)}{dg}\right)-40\left((f^{'})^3-(g^{'})^3\right)=0\end{aligned} \tag{25}$$

(here we used the identities $\partial_x=f^{'}\partial_f,\ \partial_y=g^{'}\partial_g$). Applying to (25) operator $\partial^6/\partial f^3\partial g^3$ we arrive at

$$\frac{d^6((f^{'})^3)}{df^6}-\frac{d^6((g^{'})^3)}{dg^6}=0$$

implying that $(f^{'})^3$ and $(g')^3$ are polynomials of the 6-th order in f and g, respectively. Coefficients of these polynomials are not independent and can be fixed upon substitution in (25):

$$(f^{'})^3=P(f), \quad (g^{'})^3=P(-g) \tag{26}$$

where P is an arbitrary polynomial of the 6th order. Calculations presented here follow [14], p.66-69. Formulae (26) reflect the uniformizability of Kummer's quartics by theta functions of genus 2. Since equations (23) are invariant under the transformation $(\beta,V,W)\to(-\beta,V,W)$, the class of Kummer's quartics is self-dual.

Quartics of Kummer constitute a subclass of

Isothermally asymptotic surfaces possessing a 3-parameter family of projective applicabilities which are characterized by a condition

$$(\ln\beta^2)_{xy}=c\beta^2 \tag{27}$$

for some constant c. Quartics of Kummer correspond to $c=\frac{8}{9}$. Condition (27) means that the projective metric $2\beta^2\ dxdy$ has constant Gaussian curvature $K=-c$. To investigate equations (15) with the additional constraint (27) we introduce the anzatz

$$V=(\ln\beta)_{xx}+\frac{1}{2}(\ln\beta)_x^2+A, \quad W=(\ln\beta)_{yy}+\frac{1}{2}(\ln\beta)_y^2+B, \tag{28}$$

which implies upon substitution in (15) the following equations for A,B:

$$\begin{aligned}A_y=\tfrac{3}{2}(1-c)(\beta^2)_x, \quad B_x=\tfrac{3}{2}(1-c)(\beta^2)_y,\\ (\beta^2)_xA+\beta^2A_x=(\beta^2)_yB+\beta^2B_y.\end{aligned} \tag{29}$$

For $c=1$ equations (29) are satisfied if $A=B=0$. The corresponding surfaces are improper affine spheres; they will be discussed below. Here we consider the case $c\neq1$. Introducing F by the formulae

$$A_x=-A(\ln\beta^2)_x+F, \quad B_y=-B(\ln\beta^2)_y+F \tag{30}$$

and writing down the compatibility conditions of (30) with $(29)_1$, $(29)_2$, we arrive at the equations for F

$$\begin{aligned} F_x &= 2cB\beta^2 + \tfrac{3}{2}(1-c)\tfrac{1}{\beta^2}(\beta^2(\beta^2)_y)_y \\ F_y &= 2cA\beta^2 + \tfrac{3}{2}(1-c)\tfrac{1}{\beta^2}(\beta^2(\beta^2)_x)_x \end{aligned} \tag{31}$$

the compatibility conditions of which coincide with (24). Inserting in (24) the general solution of the Liouville equation (27)

$$\beta^2 = \frac{1}{c}\frac{f^{'}g^{'}}{(f+g)^2} \tag{32}$$

we end up with the same f, g as in (26). For any β given by (32), (26) the functions V and W can be recowered from (28) where A, B, F satisfy the compatible system (29), (30), (31). Thus for any such β there exists a 3-parameter family of surfaces which have the same metric $2\beta^2\ dxdy$ and the same cubic form $\beta(dx^3 + dy^3)$ and are not projectively equivalent. In general, two projectively different surfaces in P^3 having the same projective metric and the same cubic form in a common asymptotic parametrization x, y are called projectively applicable. One can show that only for isothermally asymptotic surfaces M^2 satisfying (27) does there exist a 3-parameter family of projectively different surfaces which are all projectively applicable to M^2 (the value 3 is the maximal possible). Isothermally asymptotic surfaces possessing only one-parameter families of projective applicabilities have been investigated in [37].

The whole class of projectively applicable surfaces (not necessarily isothermally asymptotic) is discussed sect.4.

Affine spheres constitute an important subclass of isothermally asymptotic surfaces specified by the following reduction in (15):

$$V = \frac{\beta_{xx}}{\beta} - \frac{1}{2}\left(\frac{\beta_x}{\beta}\right)^2, \qquad W = \frac{\beta_{yy}}{\beta} - \frac{1}{2}\left(\frac{\beta_y}{\beta}\right)^2. \tag{33}$$

After this ansatz the first equation in (15) will be satisfied identically while the last two imply the Tzitzeica equation for β:

$$(\ln\beta)_{xy} = \beta^2 + \frac{c}{\beta}, \quad c = const. \tag{34}$$

The cases $c \neq 0$ and $c = 0$ correspond to proper and improper affine spheres, respectively. In view of (9) equations (33) and (34) are equivalent to

$$a = b = 0, \quad k = l = -\frac{c}{\beta}.$$

Using equations (8) it is easy to check that the point

$$\sqrt{\beta}\ \boldsymbol{\eta} + \frac{c}{2\sqrt{\beta}}\ \mathbf{r}$$

is independent of x, y. Geometrically, this means that all normals of Wilczynski intersect in one fixed point. This property can be viewed as a projective-invariant definition of

affine spheres (we emphasize that the normals of Wilczynski do not coincide in general with Blaschke's normals in affine differential geometry).

With V, W given by (33) equations (1) possess a particular solution $r^0 = \sqrt{\beta}$. Upon the substitution $R = r/r^0$ equations (3) assume the form

$$R_{xx} = \beta R_y - \tfrac{\beta_x}{\beta} R_x$$

$$R_{yy} = \beta R_x - \tfrac{\beta_y}{\beta} R_y$$

which become the familiar equations for the radius-vector of affine spheres after adding the compatible equation

$$R_{xy} = -\frac{c}{\beta} R.$$

The fact that Tzitzeica's equation solves the stationary mVN equation is also reflected in the following nonlocal representation of the mVN equation:

$$\beta_t = \left(\frac{1}{\beta} \partial_x \beta^2 \partial_y^{-1} \frac{1}{\beta} \partial_x - \frac{1}{\beta} \partial_y \beta^2 \partial_x^{-1} \frac{1}{\beta} \partial_y \right) \left(\beta (\ln \beta)_{xy} - \beta^3 \right).$$

Indeed, the condition $\beta(\ln \beta)_{xy} - \beta^3 = c = const$ is equivalent to 34).

We refer to [33], [6], [14], [15], [13] for the further discussion of isothermally asymptotic surfaces.

In the recent paper [29] Konopelchenko and Pinkall introduced integrable dynamics of surfaces in 3-space governed by the Veselov-Novikov equation. One can show that isothermally asymptotic surfaces can be interpreted as the stationary points of this evolution. So it is not surprising that they are described by the stationary mVN equation (which, as it has been demonstrated in [13], is equivalent to the stationary VN).

4 Projectively applicable surfaces

Generically, the quadratic form

$$\beta\gamma \, dxdy$$

and the cubic form

$$\beta \, dx^3 + \gamma \, dy^3$$

define a surface $M^2 \subset P^3$ uniquely up to projective equivalence. There exists, however, a class of the so-called projectively applicable surfaces for which this is not the case. Projective deformations of surfaces have been extensively investigated by Cartan [8]. Analytically, projective applicability means that for a given β, γ equations (2) do not specify V and W uniquely (equivalently, equations $(11)_3 - (11)_5$ do not uniquely specify a and b). As shown in [14, p. 62], projectively applicable surfaces naturally fall into two different classes: the so-called "surfaces R_0" and "surfaces R".

Surfaces $\mathbf{R_0}$ are specified by the condition $(\ln \beta)_{xy} = 0$ (or $(\ln \gamma)_{xy} = 0$), which can be reduced to $\beta = 1$ (or $\gamma = 1$) after a suitable choice of x, y. The substitution of $\beta = 1$

in (2) results in the system

$$\gamma_{xxx} - 2\gamma_x V - \gamma V_x + W_y = 0$$
$$W_x = \gamma_y \tag{35}$$
$$V_y = 2\gamma_x.$$

Since W enters (35) only through it's derivatives, it is defined up to an additive constant $W \to W + \lambda$, providing thus a 1-parameter family of projectively nonequivalent surfaces having the same β and γ. The radius-vectors of these surfaces satisfy the linear system

$$\mathbf{r}_{xx} = \mathbf{r}_y + \tfrac{1}{2}V\mathbf{r}$$
$$\mathbf{r}_{yy} = \gamma\mathbf{r}_x + \tfrac{1}{2}(W - \gamma_x + \lambda)\mathbf{r}$$

containing a "spectral parameter" λ. Rewriting this linear system in the form

$$\begin{aligned} \mathbf{r}_y &= \mathbf{r}_{xx} - \tfrac{1}{2}V\mathbf{r} \\ \mathbf{r}_{xxxx} &= V\mathbf{r}_{xx} + (V_x + \gamma)\mathbf{r}_x + \tfrac{1}{2}(V_{xx} + \gamma_x - \tfrac{1}{2}V^2 + W + \lambda)\mathbf{r} \end{aligned} \tag{36}$$

we immediately recognise in $(36)_1$ the nonstationary Schrödinger equation which plays a role of the Lax operator of the (2+1)-dimensional integrable Kadomtsev-Petviashvili (KP) hierarchy. Meanwhile, $(36)_2$ coincides with the stationary limit of the linear problem generating the second flow in the KP hierarchy.

We recall that the second flow in the KP hierarchy is associated with the linear system

$$\begin{aligned} \mathbf{r}_y &= \mathbf{r}_{xx} - \tfrac{1}{2}V\mathbf{r} \\ \mathbf{r}_t &= \mathbf{r}_{xxxx} - V\mathbf{r}_{xx} - (V_x + \gamma)\mathbf{r}_x - \tfrac{1}{2}(V_{xx} + \gamma_x - \tfrac{1}{2}V^2 + W)\mathbf{r} \end{aligned} \tag{37}$$

which is related to (36) via a formal transformation $\frac{1}{2}\lambda\mathbf{r} \to \partial_t\mathbf{r}$. The compatibility conditions of (37) result in the "second" KP flow:

$$V_t = \gamma_{xxx} - 2\gamma_x V - \gamma V_x + W_y$$
$$W_x = \gamma_y$$
$$V_y = 2\gamma_x.$$

Thus, equations (35) governing R_0-surfaces coincide with the stationary limit of the second flow in the KP hierarchy. In a sense, they can be called the higher Boussinesq equations.

Geometric properties of surfaces R_0 have been investigated, e.g., in [6], [40].

Surfaces R are specified by the condition $\beta_y = \gamma_x$. In this case V and W are defined by (2) up to additive constant $W \to W + \lambda$, $V \to V + \lambda$, so that we again arrive at

a 1-parameter family of projectively nonequivalent surfaces having the same β, γ. Their radius-vectors satisfy the linear system

$$\mathbf{r}_{xx} = \beta \mathbf{r}_y + \tfrac{1}{2}(V - \beta_y + \lambda)\mathbf{r}$$

$$\mathbf{r}_{yy} = \gamma \mathbf{r}_x + \tfrac{1}{2}(W - \gamma_x + \lambda)\mathbf{r}$$

containing a "spectral parameter" λ. Rewriting it in the form

$$\mathbf{r}_{xx} - \mathbf{r}_{yy} = \beta \mathbf{r}_y - \gamma \mathbf{r}_x + \tfrac{1}{2}(V - W)\mathbf{r}$$

$$\mathbf{r}_{xx} + \mathbf{r}_{yy} - \beta \mathbf{r}_y - \gamma \mathbf{r}_x - \tfrac{1}{2}(V + W - \beta_y - \gamma_x)\mathbf{r} = \lambda \mathbf{r},$$

we recognize the stationary DS-type linear problem (the relationship of surfaces R to the stationary DS hierarchy has been pointed out in [28]). Replacing $\lambda \mathbf{r}$ by $\partial_t \mathbf{r}$ we arrive at the (2+1)-dimensional DS linear problem

$$\mathbf{r}_{xx} - \mathbf{r}_{yy} = \beta \mathbf{r}_y - \gamma \mathbf{r}_x + \tfrac{1}{2}(V - W)\mathbf{r}$$

$$\mathbf{r}_t = \mathbf{r}_{xx} + \mathbf{r}_{yy} - \beta \mathbf{r}_y - \gamma \mathbf{r}_x - \tfrac{1}{2}(V + W - \beta_y - \gamma_x)\mathbf{r},$$

with the compatibility conditions

$$\begin{gathered} V_t + \beta_{yyy} - 2\beta_y W - \beta W_y = W_t + \gamma_{xxx} - 2\gamma_x V - \gamma V_x \\ \gamma_t = W_x - 2\gamma \beta_y - \beta \gamma_y \\ \beta_t = V_y - 2\beta \gamma_x - \gamma \beta_x \\ \beta_y = \gamma_x. \end{gathered} \tag{38}$$

In the stationary limit they reduce to the equations governing surfaces R. Introducing

$$\beta = \varphi_x, \quad \gamma = \varphi_y, \quad V = \frac{1}{2}\varphi_x^2 + \theta_{xx}, \quad W = \frac{1}{2}\varphi_y^2 + \theta_{yy},$$

we can rewrite (38) in the form

$$\varphi_t = \theta_{xy} - \varphi_x \varphi_y,$$

$$\theta_t = \varphi_{xy} - P,$$

$$\Box P = (\varphi_x \Box \theta)_y + (\varphi_y \Box \theta)_x,$$

where $\Box = \partial_x^2 - \partial_y^2$.

The coordinate net $\xi = x + y$, $\eta = x - y$ on a surface R is conjugate and has an important geometric property, namely, that both congruences of lines, tangent to the ξ- and η-coordinate curves, form a W-congruences. Conjugate nets with this property are called the R-nets; they have been introduced by Demoulin and Tzitzeica in a series of papers [9], [57]. Thus, surfaces R can be equivalently characterized by the existence

of an R-net. Bäcklund transformation for surfaces R was established by Jonas [23] and subsequently discussed in [14], [15], [11]. Some further properties of surfaces R have been studied in [18], [21]. Multidimensional analogs of surfaces R were investigated in a series of papers [31], [19], [16], [50].

The most important examples of surfaces R include:

Projective transforms of surfaces with constant Gaussian curvature $K=-1$ **in the Euclidean 3-space** which correspond to

$$\beta = -\frac{\varphi_x}{\sin\varphi}, \quad \gamma = -\frac{\varphi_y}{\sin\varphi},$$

$$V = 1 + \tfrac{1}{2}\beta^2\cos^2\varphi + (\beta\cos\varphi)_x, \qquad W = 1 + \tfrac{1}{2}\gamma^2\cos^2\varphi + (\gamma\cos\varphi)_y$$

where φ satisfies the Sine-Gordon equation $\varphi_{xy} = -\sin\varphi$. One can check directly that with this anzatz equations (2) as well as $\beta_y = \gamma_x$ are satisfied identically. Moreover, equations (1) possess a particular solution $r^0 = \frac{1}{\sqrt{\sin\varphi}}$, so that in the affine gauge $\mathbf{R} = \mathbf{r}/r^0 = \sqrt{\sin\varphi}\ \mathbf{r}$ equations (1) transform to

$$\mathbf{R}_{xx} = -\frac{\varphi_x}{\sin\varphi}\mathbf{R}_y + \frac{\varphi_x\cos\varphi}{\sin\varphi}\mathbf{R}_x$$

$$\mathbf{R}_{yy} = -\frac{\varphi_y}{\sin\varphi}\mathbf{R}_x + \frac{\varphi_y\cos\varphi}{\sin\varphi}\mathbf{R}_y.$$

Supplementing these equations with

$$\mathbf{R}_{xy} = \sin\varphi\ \mathbf{n}$$

$$\mathbf{n}_x = \frac{1}{\sin\varphi}(\mathbf{R}_y - \cos\varphi\mathbf{R}_x)$$

$$\mathbf{n}_y = \frac{1}{\sin\varphi}(\mathbf{R}_x - \cos\varphi\mathbf{R}_y)$$

(which are all mutually compatible in view of $\varphi_{xy} = -\sin\varphi$, and, moreover, possess a specialization $(\mathbf{n},\mathbf{n}) = (\mathbf{R}_x,\mathbf{R}_x) = (\mathbf{R}_y,\mathbf{R}_y) = 1,\ (\mathbf{R}_x,\mathbf{R}_y) = \cos\varphi$), we immediately recognize the equations governing the radius-vector $\mathbf{R}$ and the unit normal $\mathbf{n}$ of a surface with constant Gaussian curvature $K=-1$ parametrized by asymptotic coordinates x,y. It's first and second fundamental forms are given by the formulae

$$dx^2 + 2\cos\varphi\ dxdy + dy^2$$

$$2\sin\varphi\ dxdy,$$

respectively. In this case the conjugate net R is the net of curvature lines.

Closely related examples of surfaces R are focal surfaces of the congruence of normals of a surface with $K=-1$.

Projective transforms of surfaces with constant Gaussian curvature $K=1$ **in the Lorentzian 3-space** are characterized by exactly the same formulae as surfaces with $K=-1$ in the Euclidean space with the interchange $\sin\to\sinh,\ \cos\to\cosh$ in all

places where they occur. These surfaces are described by the equation $\varphi_{xy} = -\sinh\varphi$ and have the fundamental forms

$$dx^2 + 2\cosh\varphi\; dxdy + dy^2$$

$$2\sinh\varphi\; dxdy,$$

respectively. Note that the first fundamental form is now indefinite.

Surfaces with an R-net of period 4 are characterized by

$$\beta = \tfrac{1}{2}\varphi_x, \quad \gamma = \tfrac{1}{2}\varphi_y$$

$$V = \tfrac{1}{4}\varphi_x^2 + \tfrac{1}{8}\varphi_y^2 + \sinh\varphi, \qquad W = \tfrac{1}{4}\varphi_y^2 + \tfrac{1}{8}\varphi_x^2 - \sinh\varphi$$

where $\varphi_{xx} - \varphi_{yy} = 4\cosh\varphi$. With these β, γ, V, W equations (2) and $\beta_y = \gamma_x$ are satisfied identically. One can show that under Laplace transformations the R-net $\xi = x + y,\ \eta = x - y$ on these surfaces is periodic with period four. Moreover, all surfaces carrying an R-net of period four can be obtained in this way.

5 Surfaces of Jonas

are characterized by a condition

$$\beta_x = \gamma_y$$

which is equivalent to the requirement that the conjugate net $\xi = x + y,\ \eta = x - y$ on the surface M^2 has equal point and tangential Laplace invariants (tangential Laplace invariants are the Laplace invariants of the conjugate net ξ, η on the dual surface). This class of surfaces has been introduced by Jonas in [24] as a counterpart of surfaces R and subsequently discussed in [7], [18], [43] as well as in the textbooks cited above. Conjugate nets with equal point and tangential Laplace invariants are called the nets of Jonas. The condition $\beta_x = \gamma_y$ allows an introduction of a spectral parameter λ in the linear system (13). For that purpose we first define H and K by the formulae

$$\begin{aligned} H_x &= -\gamma\ K + \lambda\ \mathcal{V}, \qquad H_y = -\beta\ K \\ K_y &= -\beta\ H + \lambda\ \mathcal{U}, \qquad K_x = -\gamma\ H \end{aligned} \tag{39}$$

which are compatible in view of (13) and $\beta_x = \gamma_y$. The systems (13) and (39) can be coupled in an 8×8 linear system

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_x}{\gamma} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 1 & 0 & 0 \\ -\beta a & 0 & \beta & 0 & b & -\frac{\gamma_x}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\gamma b & 0 & \gamma & 0 & 1 \\ \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta \\ \lambda & 0 & 0 & 0 & 0 & 0 & -\beta & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix} \tag{40}$$

The coupling is achieved by adding K to $\mathcal{P}_y$ and H to $\mathcal{Q}_x$. Linear system (40) reduces to (13) under the reduction $\lambda = H = K = 0$. This linear system is gauge equivalent to the one used in [24] for the construction of the Bäcklund transformation for the Jonas surfaces. With $\beta = \varphi_y,\ \gamma = \varphi_x$ equations (2) assume the form

$$\begin{gathered} \varphi_{yyyy} - 2\ \varphi_{yy}\ W - \varphi_y\ W_y = \varphi_{xxxx} - 2\ \varphi_{xx}\ V - \varphi_x\ V_x \\ (W + \tfrac{1}{2}\varphi_y^2)_x = 2\ (\varphi_x \varphi_y)_y \\ (V + \tfrac{1}{2}\varphi_x^2)_y = 2\ (\varphi_x \varphi_y)_x. \end{gathered} \tag{41}$$

Rewriting equations (40) in terms of $\mathcal{U}, \mathcal{V}$ (that is, expressing $\mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{Q}, H, K$ through $\mathcal{U}, \mathcal{U}_y, \mathcal{U}_{yy}, \mathcal{U}_{yyy}$ and $\mathcal{V}, \mathcal{V}_x, \mathcal{V}_{xx}, \mathcal{V}_{xxx}$), we arrive at the equivalent spectral problem of the system (41):

$$\begin{aligned} \mathcal{U}_x &= \varphi_y\ \mathcal{V} \\ \mathcal{V}_y &= \varphi_x\ \mathcal{U} \end{aligned} \tag{42}$$

$$\begin{gathered} \lambda\, \mathcal{U} = \mathcal{U}_{xxxx} + \mathcal{U}_{yyyy} - (\varphi_y^2 + 2W)\ \mathcal{U}_{yy} - 4\ \varphi_{xy}\ \mathcal{V}_{xx} \\ +(\varphi_y\ \varphi_{yy} - 3\ W_y)\ \mathcal{U}_y - (2\ \varphi_{xxy} + \varphi_y\ \varphi_x^2 + 2\ V\varphi_y)\ \mathcal{V}_x + m\ \mathcal{U} + n\ \mathcal{V}, \\ \lambda\ \mathcal{V} = \mathcal{V}_{xxxx} + \mathcal{V}_{yyyy} - (\varphi_x^2 + 2V)\ \mathcal{V}_{xx} - 4\ \varphi_{xy}\ \mathcal{U}_{yy} \\ +(\varphi_x\ \varphi_{xx} - 3\ V_x)\ \mathcal{V}_x - (2\ \varphi_{xyy} + \varphi_x\ \varphi_y^2 + 2\ W\varphi_x)\ \mathcal{U}_y + \tilde{m}\ \mathcal{V} + \tilde{n}\ \mathcal{U}, \end{gathered} \tag{43}$$

where $\lambda = const$ and $m,\ \tilde{m},\ n,\ \tilde{n}$ are given by the formulae

$$m = -W_{yy} - \varphi_y\ \varphi_{yyy} - 2\ \varphi_x\ \varphi_{xxx} + \varphi_{xx}^2 + 2\ V\ \varphi_x^2 + 2\ W\ \varphi_y^2$$

$$\tilde{m} = -V_{xx} - \varphi_x\ \varphi_{xxx} - 2\ \varphi_y\ \varphi_{yyy} + \varphi_{yy}^2 + 2\ V\ \varphi_x^2 + 2\ W\ \varphi_y^2$$

$$n = -2\ \varphi_{xxxy} + \varphi_x^2\ \varphi_{xy} + 3\ \varphi_x\ \varphi_y\ \varphi_{xx} + 2\ V\ \varphi_{xy} - \varphi_y\ V_x$$

$$\tilde{n} = -2\ \varphi_{xyyy} + \varphi_y^2\ \varphi_{xy} + 3\ \varphi_x\ \varphi_y\ \varphi_{yy} + 2\ W\ \varphi_{xy} - \varphi_x\ W_y.$$

The structure of linear system (42), (43) clearly suggests that equations (41) are related to the stationary limit of the fourth order flow in the DS hierarchy. Indeed, the fourth order flow in the DS hierarchy is generated by the 2-dimensional Dirac operator

$$\begin{aligned} \mathcal{U}_x &= \beta\ \mathcal{V} \\ \mathcal{V}_y &= \gamma\ \mathcal{U} \end{aligned} \tag{44}$$

where the time evolution of $\mathcal{U}$ and $\mathcal{V}$ is specified by

$$\begin{aligned} \mathcal{U}_t &= \mathcal{U}_{xxxx} + \mathcal{U}_{yyyy} + ... \\ \mathcal{V}_t &= \mathcal{V}_{xxxx} + \mathcal{V}_{yyyy} + ... \end{aligned} \tag{45}$$

The linear problem corresponding to the stationary flow can be obtained by a formal substitution $\mathcal{U}_t \to \lambda\mathcal{U},\ \mathcal{V}_t \to \lambda\mathcal{V}$ where λ plays a role of spectral parameter. This linear problem reduces to (42), (43) after the additional reduction $\beta_x = \gamma_y$. We emphasize that the reduction $\beta_x = \gamma_y$ is compatible not with the (2+1)-dimensional flow generated by (44), (45), rather than only with its stationary limit. This reduction is compatible only with the odd order flows in the DS hierarchy.

Particular examples of the Jonas surfaces are provided by

Projective transforms of minimal surfaces in the Euclidean 3-space corresponding to

$$\beta = \varphi_y, \quad \gamma = \varphi_x,$$

$$V = \tfrac{1}{2}\varphi_x^2 - \varphi_y^2 - \exp 2\varphi, \quad W = \tfrac{1}{2}\varphi_y^2 - \varphi_x^2 - \exp 2\varphi$$

where φ satisfies the Liouville equation $\varphi_{xx} + \varphi_{yy} = -\exp 2\varphi$. One can check directly that with this anzatz equations (2) as well as $\beta_x = \gamma_y$ are satisfied identically. Moreover, equations (1) possess a particular solution $r^0 = \exp\frac{\varphi}{2}$, so that in the affine gauge $\mathbf{R} = \mathbf{r}/r^0 = \exp(-\frac{\varphi}{2})\mathbf{r}$ equations (1) transform to

$$\begin{aligned} \mathbf{R}_{xx} &= \varphi_y\ \mathbf{R}_y - \varphi_x\ \mathbf{R}_x \\ \mathbf{R}_{yy} &= \varphi_x\ \mathbf{R}_x - \varphi_y\ \mathbf{R}_y. \end{aligned}$$

Supplementing these equations with

$$\mathbf{R}_{xy} = -\varphi_y\ \mathbf{R}_x - \varphi_x\ \mathbf{R}_y - \mathbf{n}$$

$$\mathbf{n}_x = \exp 2\varphi\ \mathbf{R}_y$$

$$\mathbf{n}_y = \exp 2\varphi\ \mathbf{R}_x$$

which are all mutually compatible in view of $\varphi_{xx}+\varphi_{yy}=-\exp 2\varphi$, and, moreover, possess a specialization $(\mathbf{n},\mathbf{n})=1,\ (\mathbf{R}_x,\mathbf{R}_x)=(\mathbf{R}_y,\mathbf{R}_y)=\exp(-2\varphi),\ (\mathbf{R}_x,\mathbf{R}_y)=0$, we immediately recognize the equations governing the radius-vector $\mathbf{R}$ and the unit normal $\mathbf{n}$ of a minimal surface parametrized by asymptotic coordinates x,y. It's first and second fundamental forms are given by the formulae

$$\exp(-2\varphi)(dx^2+dy^2)$$

$$-2\ dxdy,$$

respectively. For minimal surfaces the conjugate net of Jonas is the net of curvature lines.

In [24] there was established a one-to-one correspondence between surfaces of Jonas and pairs of surfaces in the Euclidean 3-space such that:
– both surfaces are in isometric correspondence;
– this correspondence preserves a congugate net with equal point Laplace invariants.
In a sense, this construction is an analog of the correspondence between isothermic surfaces and Bonnet pairs (Bonnet pairs are pairs of surfaces in isometric correspondence such that the mean curvature in the corresponding points is the same).

6 Projectively minimal surfaces

These surfaces are the extremals of the projective area functional

$$\int\int \beta\gamma\, dxdy. \tag{46}$$

The Euler-Lagrange equations for the functional (46) have been derived in [53], [45] and in our notation adopt the form

$$\beta_{yyy}-2\beta_y W-\beta W_y=0, \qquad \gamma_{xxx}-2\gamma_x V-\gamma V_x=0$$

$$W_x=2\gamma\beta_y+\beta\gamma_y$$

$$V_y=2\beta\gamma_x+\gamma\beta_x$$

which results after equating to zero both sides of $(2)_1$. Exploiting the obvious symmetry $\beta\to\lambda\beta,\ \gamma\to\frac{1}{\lambda}\gamma$, we can introduce the spectral parameter in the linear system (1):

$$\mathbf{r}_{xx}=\lambda\ \beta\ \mathbf{r}_y+\tfrac{1}{2}(V-\lambda\ \beta_y)\ \mathbf{r}$$

$$\mathbf{r}_{yy}=\tfrac{1}{\lambda}\ \gamma\ \mathbf{r}_x+\tfrac{1}{2}(W-\tfrac{1}{\lambda}\ \gamma_x)\ \mathbf{r}.$$

In terms of (11) the Euler-Lagrange equations reduce to

$$\beta a_y+2a\beta_y=0, \quad \gamma b_x+2b\gamma_x=0,$$

which result after equating to zero both sides of equation $(11)_5$. In the Plücker coordinates the spectral problem assumes the form

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \lambda\beta & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\lambda\beta a & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_x}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ -\lambda\beta a & 0 & \lambda\beta & 0 & b & -\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\frac{\gamma b}{\lambda} & 0 & \frac{\gamma}{\lambda} \\ \frac{\gamma}{\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -\frac{\gamma b}{\lambda} & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix} \tag{47}$$

Setting

$$a = \frac{\varphi(x)}{\beta^2}, \quad b = \frac{\psi(y)}{\gamma^2},$$

we have three cases to distinguish:

General case. Both $\varphi(x)$ and $\psi(y)$ are nonzero. In this case, we can always normalize $\varphi(x), \psi(y)$ to ± 1 by means of transformations (6). Let us assume, for instance, that $\varphi(x) = \psi(y) = 1$. With this normalization equations (11) assume the form

$$\begin{aligned} (\ln\beta)_{xy} = \beta\gamma - k, \quad & (\ln\gamma)_{xy} = \beta\gamma - l, \\ (\beta k)_y + 2\tfrac{\beta_x}{\beta^2} = 0, \quad & (\gamma l)_x + 2\tfrac{\gamma_y}{\gamma^2} = 0. \end{aligned} \tag{48}$$

Surfaces of Godeaux-Rozet [6, p. 318]. In this case, $\varphi = 0$, and hence $a = 0$, while ψ is nonzero and may be normalized to ± 1. Here, we assume that $\psi = 1$. Inserting this ansatz in (11) we obtain

$$k = \frac{s(x)}{\beta}$$

Hence, if $s(x)$ is nonzero, it may be reduced to -1 by means of (6) so that the resulting equations take the form

$$\begin{gathered} (\ln\beta)_{xy} = \beta\gamma + \tfrac{1}{\beta}, \quad (\ln\gamma)_{xy} = \beta\gamma - l, \\ (\gamma l)_x + 2\tfrac{\gamma_y}{\gamma^2} = 0. \end{gathered} \tag{49}$$

Surfaces of Demoulin. In this case, both φ and ψ are zero and hence $a = b = 0$, so that

$$k = \frac{s(x)}{\beta}, \quad l = \frac{t(y)}{\gamma}.$$

Once again, the analysis falls into three subcases depending on whether s, t are zero or not. In the generic situation $s \neq 0,\ t \neq 0$ both s and t may be normalized to -1 and the resulting equations reduce to the coupled Tzitzeica system

$$
\begin{aligned}
(\ln \beta)_{xy} &= \beta\gamma + \tfrac{1}{\beta}, \\
(\ln \gamma)_{xy} &= \beta\gamma + \tfrac{1}{\gamma},
\end{aligned}
\tag{50}
$$

In this form, the equations governing Demoulin surfaces have been set down in [14, p. 51]. The same system has been presented in [41] as a reduction of the two-dimensional Toda lattice. Inserting $a = b = 0,\ k = -\frac{1}{\beta},\ l = -\frac{1}{\gamma}$ in equations (47) and rewriting them in terms of $\mathcal{A}$ and $\mathcal{B}$, we arrive at a more convenient second-order linear problem [12]

$$
\begin{aligned}
\mathcal{B}_{xy} &= -\tfrac{1}{\gamma}\ \mathcal{B} & \qquad \mathcal{A}_{xy} &= -\tfrac{1}{\beta}\ \mathcal{A} \\
\mathcal{B}_{xx} &= \lambda\ \beta\ \mathcal{A}_y - \tfrac{\gamma_x}{\gamma}\ \mathcal{B}_x & \qquad \mathcal{A}_{xx} &= \lambda\ \gamma\ \mathcal{B}_y - \tfrac{\beta_x}{\beta}\ \mathcal{A}_x \\
\mathcal{B}_{yy} &= \tfrac{\beta}{\lambda}\ \mathcal{A}_x - \tfrac{\gamma_y}{\gamma}\ \mathcal{B}_y & \qquad \mathcal{A}_{yy} &= \tfrac{\gamma}{\lambda}\ \mathcal{B}_x - \tfrac{\beta_y}{\beta}\ \mathcal{A}_y.
\end{aligned}
\tag{51}
$$

In [10] Demoulin established in a purely geometric manner the existence of Bäcklund transformations for Godeaux-Rozet and Demoulin surfaces and proved the corresponding permutability theorems. Apparently, Demoulin did not formulate his results in terms of analytic expressions. In [12], a Toda lattice connection and the second-order linear problem (51) is used to derive explicitly a Bäcklund transformation for Demoulin surfaces.

Remark. The specialization $\beta = \gamma$ reduces (50) to the Tzitzeica equation

$$
(\ln \beta)_{xy} = \beta^2 + \frac{1}{\beta},
$$

which governs affine spheres in affine differential geometry [56]. Moreover, with $\mathcal{A} = \mathcal{B}$ the linear system (51) becomes the standard Lax pair for the Tzitzeica equation. Geometrically this means that affine spheres lie in the intersection of two different integrable classes of projective surfaces, namely isothermally asymptotic and projectively minimal surfaces.

Projectively minimal, Godeaux-Rozet and Demoulin surfaces also arise in the theory of envelopes of Lie quadrics associated with the surface M^2. For brevity, we only recall the necessary definitions. The details can be found in [6], [14], [33], etc. Thus, let us consider a point p^0 on the surface M^2 and the x-asymptotic line passing through p^0. Let us take three additional points p^i, $i = 1, 2, 3$ on this asymptotic line close to p^0 and draw three y-asymptotic lines γ^i passing through p^i. The three straight lines which are tangential to γ^i and pass through the points p^i uniquely define a quadric $\mathbf{Q}$ containing them as rectilinear generators. As p^i tend to p^0, the quadric $\mathbf{Q}$ tends to a limiting quadric, the so-called Lie quadric of the surface M^2 at the point p^0. Even though this construction depends on the initial choice of either the x- or the y-asymptotic line through p^0, the resulting quadric $\mathbf{Q}$ is independent of that choice. Thus, we arrive at a two-parameter family of quadrics associated with the surface M^2. In terms of the Wilczynski tetrahedral, the parametric equation for $\mathbf{Q}$ is of the form [6, p.311]

$$
\mathbf{Q} = \boldsymbol{\eta} + \mu \mathbf{r}_1 + \nu \mathbf{r}_2 + \mu\nu \mathbf{r},
$$

where μ, ν are parameters.

Now, in the neighbourhood of a generic point p^0 on M^2, the envelopes of the family of Lie quadrics consist of the surface M^2 itself and four, in general, distinct sheets. The case of projectively minimal surfaces is characterized by the additional requirement that the asymptotic lines on all these sheets correspond to the asymptotic lines of the surface M^2 itself [54]. Moreover, for projectively minimal surface all four sheets of the envelope will be projectively minimal as well [39]. In a sense, it is natural to call the family of Lie quadrics with this property a W-congruence of quadrics. Surfaces of Godeaux-Rozet are characterized by the degenerate case of two distinct sheets while Demoulin surfaces are present if all four sheets coincide. Surfaces of Godeaux-Rozet and Demoulin have been investigated extensively in [10, 20, 44], see also [12].

7 Congruences W

There exists an important class of transformations in projective differential geometry which leave the system (1) form-invariant. These are transformations generated by congruences W. Here we briefly recall this construction following [23], [14], [11].

Let M^2 be a surface with the radius-vector $\mathbf{r}$ satisfying (1). Let the functions $\mathcal{U}$ and $\mathcal{V}$ satisfy the Dirac equation

$$\begin{aligned} \mathcal{U}_x &= \beta\, \mathcal{V} \\ \mathcal{V}_y &= \gamma\, \mathcal{U} \end{aligned} \tag{52}$$

where β and γ are the same as in (1). Let us consider a surface $\tilde{M}^2$ with the radius-vector $\mathbf{r}'$ given by the formula

$$\mathbf{r}' = \mathcal{V}\ \mathbf{r}_1 - \mathcal{U}\ \mathbf{r}_2 + \frac{1}{2}\left(\mathcal{V}\ \frac{\gamma_x}{\gamma} - \mathcal{U}\ \frac{\beta_y}{\beta} - \mathcal{V}_x + \mathcal{U}_y\right)\ \mathbf{r} \tag{53}$$

In order to write down the equations for $\mathbf{r}'$ it is convenient to introduce certain quantities which are combinations of $\mathcal{U}, \mathcal{V}$ and their derivatives. First of all, we define $\mathcal{A}$ and $\mathcal{B}$ by the formulae

$$\mathcal{U}_y = \frac{\beta_y}{\beta}\ \mathcal{U} + \mathcal{A}, \qquad \mathcal{V}_x = \frac{\gamma_x}{\gamma}\ \mathcal{V} + \mathcal{B},$$

(in fact, we are copying equations (13) for the Plücker coordinates). The compatibility conditions $\mathcal{U}_{xy} = \mathcal{U}_{yx}$ and $\mathcal{V}_{xy} = \mathcal{V}_{yx}$ imply

$$\mathcal{A}_x = k\ \mathcal{U}, \qquad \mathcal{B}_y = l\ \mathcal{V},$$

where l and k are the same as in (9). Let us introduce $\mathcal{P}$ and $\mathcal{Q}$ by the formulae

$$\mathcal{A}_y = a\ \mathcal{U} + \mathcal{P}, \qquad \mathcal{B}_x = b\ \mathcal{V} + \mathcal{Q}.$$

Then the compatibility conditions imply

$$\mathcal{P}_x = -\beta a\ \mathcal{V} + k\ \mathcal{A}, \qquad \mathcal{Q}_y = -\gamma b\ \mathcal{U} + l\ \mathcal{B}.$$

Finally, we introduce the quantities H and K via

$$\mathcal{P}_y = a\ \mathcal{A} - \frac{\beta_y}{\beta}\ \mathcal{P} - \gamma b\ \mathcal{V} + \gamma\ \mathcal{Q} + K, \qquad \mathcal{Q}_x = b\ \mathcal{B} - \frac{\gamma_x}{\gamma}\ \mathcal{Q} - \beta a\ \mathcal{U} + \beta\ \mathcal{P} + H,$$

so that the compatibility conditions imply

$$H_y = -\beta\ K, \qquad K_x = -\gamma\ H.$$

Equations for $\mathcal{U}, \mathcal{A}, \mathcal{P}, \mathcal{V}, \mathcal{B}, \mathcal{Q}, H, K$ can be rewritten in the matrix form

$$\begin{gathered}
\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_x}{\gamma} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 1 & 0 & 0 \\ -\beta a & 0 & \beta & 0 & b & -\frac{\gamma_x}{\gamma} & 1 & 0 \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix} \\
\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\gamma b & 0 & \gamma & 0 & 1 \\ \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta \\ * & * & * & * & * & * & * & * \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}
\end{gathered} \tag{54}$$

where the elements $*$ are not yet specified. Equations (54) reduce to (13) under the reduction $H = K = 0$. In what follows we will also need the quantity

$$S = \mathcal{Q}\ \mathcal{V} - \mathcal{P}\ \mathcal{U} + \frac{\mathcal{A}^2 - \mathcal{B}^2}{2},$$

which, in view of (54), satisfies the equations

$$S_x = H\ \mathcal{V}, \qquad S_y = -K\ \mathcal{U}.$$

Now a direct calculation gives:

$$\mathbf{r} = -2\frac{\mathcal{V}}{S}\ \mathbf{r}'_x - 2\frac{\mathcal{U}}{S}\ \mathbf{r}'_y + \frac{1}{S}(\mathcal{A} + \mathcal{B} + \frac{\gamma_x}{\gamma}\mathcal{V} + \frac{\beta_y}{\beta}\mathcal{U})\ \mathbf{r}'. \tag{55}$$

Equations (53) and (55) immediately imply that the line $\mathbf{r} \wedge \mathbf{r}'$ joining the corresponding points $\mathbf{r}$ and $\mathbf{r}'$ is tangent to both surfaces M^2 amd $\tilde{M}^2$, which are thus the focal surfaces of the line congruence $\mathbf{r} \wedge \mathbf{r}'$. Moreover, the formulae

$$\begin{aligned}
\mathbf{r}'_{xx} &= \tfrac{S_x}{S}\ \mathbf{r}'_x + \left(\tfrac{S_x}{S}\tfrac{\mathcal{U}}{\mathcal{V}} - \beta\right)\ \mathbf{r}'_y + \tfrac{1}{2}\left(V + \beta_y - \tfrac{S_x}{S\mathcal{V}}(\mathcal{A} + \mathcal{B} + \tfrac{\gamma_x}{\gamma}\mathcal{V} + \tfrac{\beta_y}{\beta}\mathcal{U})\right)\mathbf{r}', \\
\mathbf{r}'_{yy} &= \tfrac{S_y}{S}\ \mathbf{r}'_y + \left(\tfrac{S_y}{S}\tfrac{\mathcal{V}}{\mathcal{U}} - \gamma\right)\ \mathbf{r}'_x + \tfrac{1}{2}\left(W + \gamma_x - \tfrac{S_y}{S\mathcal{U}}(\mathcal{A} + \mathcal{B} + \tfrac{\gamma_x}{\gamma}\mathcal{V} + \tfrac{\beta_y}{\beta}\mathcal{U})\right)\mathbf{r}',
\end{aligned} \tag{56}$$

(which are the result of quite a long calculation) demonstrate that x, y are asymptotic coordinates on the transformed surface $\tilde{M}^2$ as well, so that the congruence $\mathbf{r} \wedge \mathbf{r}'$ preserves

the asymptotic parametrization of it's focal surfaces. Such congruences play a central role in projective differential geometry and are known as the congruences W. By a construction, a congruence W with one given focal surface M^2 is uniquely determined by a solution $\mathcal{U}, \mathcal{V}$ of the linear Dirac equation (52). Normalising the vector $\mathbf{r}'$ as follows: $\mathbf{r}' = \sqrt{S}\ \tilde{\mathbf{r}}$ we can rewrite equations (56) in the canonical form (1) in terms of $\tilde{\mathbf{r}}$:

$$
\begin{aligned}
\tilde{\mathbf{r}}_{xx} &= \tilde{\beta}\ \tilde{\mathbf{r}}_y + \tfrac{1}{2}(\tilde{V} - \tilde{\beta}_y)\ \tilde{\mathbf{r}} \\
\tilde{\mathbf{r}}_{yy} &= \tilde{\gamma}\ \tilde{\mathbf{r}}_x + \tfrac{1}{2}(\tilde{W} - \tilde{\gamma}_x)\ \tilde{\mathbf{r}}
\end{aligned}
\tag{57}
$$

where the transformed coefficients $\tilde{\beta},\ \tilde{\gamma},\ \tilde{V},\ \tilde{W}$ are given by the formulae

$$
\begin{aligned}
\tilde{\beta} &= \tfrac{S_x}{S}\tfrac{\mathcal{U}}{\mathcal{V}} - \beta = \tfrac{H\mathcal{U}}{S} - \beta, \qquad & \tilde{\gamma} &= \tfrac{S_y}{S}\tfrac{\mathcal{V}}{\mathcal{U}} - \gamma = -\tfrac{K\mathcal{V}}{S} - \gamma, \\
\tilde{V} &= V - \tfrac{S_x}{S}\tfrac{\mathcal{V}_x}{\mathcal{V}} + \tfrac{3}{2}(\tfrac{S_x}{S})^2 - \tfrac{S_{xx}}{S}, \qquad & \tilde{W} &= W - \tfrac{S_y}{S}\tfrac{\mathcal{U}_y}{\mathcal{U}} + \tfrac{3}{2}(\tfrac{S_y}{S})^2 - \tfrac{S_{yy}}{S},
\end{aligned}
\tag{58}
$$

(we point out the usefull identity $\tilde{\beta}\tilde{\gamma} = \beta\gamma - (\ln S)_{xy}$). We will call the surface $\tilde{M}^2$ a W-transform of the surface M^2. Congruences W provide a standard tool for constructing Bäcklund transformations. Suppose we are given a class of surfaces specified by a certain constraint between $\beta,\ \gamma,\ V,\ W$. Let us try to find a congruence W such that the second focal surface will also belong to the same class. This requirement imposes additional restrictions on the functions $\mathcal{U}$ and $\mathcal{V}$, which usually turn to be linear and, moreover, contain an arbitrary constant parameter, so that equations (54) become a "Lax pair" for the class of surfaces under study. Since the Dirac equation (52) is a part of this Lax pair, it is not surprising that surfaces in projective geometry are closely related to the DS hierarchy. Particularly interesting classes of surfaces correspond to reductions of the Dirac operator which are quite familiar from the modern soliton theory. These are

isothermally-asymptotic surfaces ($\beta = \gamma$);
surfaces R_0 ($\beta = 1$ or $\gamma = 1$),
surfaces R ($\beta_y = \gamma_x$);
surfaces of Jonas ($\beta_x = \gamma_y$), etc.

Example 1. Bäcklund transformation for isothermally asymptotic surfaces. Let us require that both surfaces M^2 and $\tilde{M}^2$ are isothermally asymptotic, that is, $\beta = \gamma$ and $\tilde{\beta} = \tilde{\gamma}$. For that purpose it is sufficient to choose

$$H = \lambda\ \mathcal{V}, \qquad K = -\lambda\ \mathcal{U}, \qquad \lambda = const,$$

which, upon the substitution in (54), results in the Lax pair (16). The transformation

$$\tilde{\beta} = \lambda\frac{\mathcal{U}\mathcal{V}}{S} - \beta$$

provides thus a Bäcklund transformation of the stationary mVN equation.

Example 2. Bäcklund transformation for surfaces R_0. Here we require $\beta = \tilde{\beta} = 1$, implying $H\mathcal{U} = 2S$. Differentiation by x and y produces further constraints $H_x\mathcal{U} = H\mathcal{V}$ and $K\mathcal{U} + H\mathcal{A} = 0$, respectively. These restrictions can be identically satisfied if we assume

$$H = -\lambda\ \mathcal{U}, \qquad K = \lambda\ \mathcal{A},$$

and impose the compatible quadratic constraint

$$\lambda\, \mathcal{U}^2 + 2\ S = 0.$$

Example 3. Bäcklund transformation for surfaces R. Here we require $\beta_y = \gamma_x$ and $\tilde\beta_y = \tilde\gamma_x$ implying $(H\mathcal{U}/S)_y + (K\mathcal{V}/S)_x = 0$. To satisfy this condition, it is sufficient to choose

$$H = \lambda \left(\frac{\gamma_x}{\gamma}\ \mathcal{V} + \mathcal{B} - \beta\, \mathcal{U} \right), \qquad K = \lambda \left(\frac{\beta_y}{\beta}\ \mathcal{U} + \mathcal{A} - \gamma\ \mathcal{V} \right)$$

and to impose the compatible quadratic constraint

$$\lambda\ \left(\mathcal{U}^2 - \mathcal{V}^2\right) + 2\ S = 0.$$

In a different gauge this Bäcklund transformation has been set down by Jonas in [23].

Example 4. Bäcklund transformation for surfaces of Jonas. Here we require $\beta_x = \gamma_y$ and $\tilde\beta_x = \tilde\gamma_y$ implying $(H\mathcal{U}/S)_x + (K\mathcal{V}/S)_y = 0$. This restiction is satisfied if H and K satisfy the equations

$$H_x = \lambda\ \mathcal{V} - \gamma\ K, \qquad K_y = \lambda\ \mathcal{U} - \beta\ H,$$

along with the additional quadratic constraint

$$2\lambda\ S + K^2 - H^2 = 0.$$

With these H, K equations (54) transform into the 8×8 linear problem (40) which is compatible with the above quadratic constraint. This Bäcklund transformation has been set down by Jonas in [24].

Example 5. Bäcklund transformation for surfaces with one family of asymptotic lines in linear complexes. This class of surfaces is specified by the condition $k = 0$ in equations (11), implying $(\ln\beta)_{xy} = \beta\gamma$. Geometrically, this means that the Plücker image $\mathcal{U} = \mathbf{r}\wedge\mathbf{r}_1$ of any x-asymptotic curve of the surface M^2 is a hyperplane curve in P^5 (one can similarly require $l = 0$ or $(\ln\gamma)_{xy} = \beta\gamma$ implying the same property for y-asymptotic curves). Indeed, the condition $k = 0$ implies $\mathcal{A}_x = 0$ (see (13)), so that the vector $\mathcal{A}$ is constant along any x-asymptotic line. The conditions $(\mathcal{A}, \mathcal{U}) = (\mathcal{A}, \mathcal{U}_x) = ... = 0$ (see (14)) means that the curve $\mathcal{U}$ lies in a hyperplane in P^5 which is orthogonal to $\mathcal{A}$. We recall, that by a definition a linear complex is a 3-parameter family of straight lines in P^3, which corresponds to a hyperplane in P^5 under the Plücker embedding. Surfaces with asymptotic lines in linear complexes have been extensively studied in projective differential geometry, see e.g. [25], [26], [15] [17] and references therein. The most important geometric property of these surfaces is the existence of a W-congruence, mapping a surface M^2 onto a ruled surface $\tilde M^2$ for which $\tilde\beta = 0$. In this case the x-asymptotic curves of the transformed surface are just straight lines. This construction rectifies all x-asymptotic curves of the surface M^2 simultaneosly and in fact linearises equations (11) with the constraint $k = 0$. Requiring in (58) $\tilde\beta = 0$, or, equivalently, $H\mathcal{U} = \beta S$, and differentiating this constraint with respect to x and y, we arrive at the additional constraints

$$H_x = \frac{\beta_x}{\beta} H \qquad \text{and} \qquad \mathcal{A} = 0,$$

respectively. Taking into account (54), the last condition implies

$$\mathcal{P} = -a\,\mathcal{U}, \qquad K = -(a_y + 2a\frac{\beta_y}{\beta})\,\mathcal{U} + \gamma b\,\mathcal{V} - \gamma\,\mathcal{Q},$$

so that equations (54) reduce to

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \end{pmatrix}_x = \begin{pmatrix} 0 & \beta & 0 & 0 & 0 \\ 0 & \frac{\gamma_x}{\gamma} & 1 & 0 & 0 \\ 0 & b & 0 & 1 & 0 \\ -2\beta a & 0 & b & -\frac{\gamma_x}{\gamma} & 1 \\ 0 & 0 & 0 & 0 & \frac{\beta_x}{\beta} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 \\ 0 & l & 0 & 0 & 0 \\ -\gamma b & 0 & l & 0 & 0 \\ \beta a_y + 2a\beta_y & -\beta\gamma b & 0 & \beta\gamma & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \end{pmatrix}$$

This system is compatible and preserves the constraint $H\mathcal{U} = \beta S$. Thus, it defines a 3-parameter family of transformations W mapping a surface M^2 with x-asymptotic lines in linear complexes onto a ruled surface $\tilde{M}^2$. Conversely, applying an arbitrary transformation W to a ruled surface, we obtain a surface with one family of asymptotic lines in linear complexes (those corresponding to rectilinear generators of a ruled surface).

Example 5. Bäcklund transformation for surfaces with both families of asymptotic lines in linear complexes. This class of surfaces is specified by the condition $k = l = 0$ implying a pair of the Liouville equations for β, γ:

$$(\ln\beta)_{xy} = \beta\gamma, \qquad (\ln\gamma)_{xy} = \beta\gamma.$$

In this case one can find a transformation W, which maps M^2 onto a surface $\tilde{M}^2$ for which $\tilde{\beta} = \tilde{\gamma} = 0$, that is, onto a quadric. Thus, it simultaneously rectifies all asymptotic curves. In order to construct such a W-transformation we have to take

$$\mathcal{A} = \mathcal{B} = 0, \qquad \mathcal{P} = -a\,\mathcal{U}, \qquad \mathcal{Q} = -b\,\mathcal{V},$$

$$H = -(b_x + 2b\frac{\gamma_x}{\gamma})\,\mathcal{V} + 2\beta a\,\mathcal{U}, \qquad K = -(a_y + 2a\frac{\beta_y}{\beta})\,\mathcal{U} + 2\gamma b\,\mathcal{V},$$

implying the following equations for $\mathcal{U}, \mathcal{V}$

$$\mathcal{U}_x = \beta\,\mathcal{V}, \quad \mathcal{U}_y = \frac{\beta_y}{\beta}\,\mathcal{U}, \quad \mathcal{V}_x = \frac{\gamma_x}{\gamma}\,\mathcal{V}, \quad \mathcal{V}_y = \gamma\,\mathcal{U}.$$

together with the compatible quadratic constraint

$$\beta\gamma\ (a\,\mathcal{U}^2 + b\,\mathcal{V}^2) = (\gamma b_x + 2b\gamma_x)\,\mathcal{U}\mathcal{V}.$$

These $\mathcal{U}, \mathcal{V}$ define a unique transformation W, mapping our surface to a quadric. Conversely, applying an arbitrary W-transformation to a quadric, we obtain a surface with both families of asymptotic lines in linear complexes.

8 Acknowledgements

I would like to thank A.I. Bobenko, K.R. Khusnutdinova, B.G. Konopelchenko, U. Pinkall and W.K. Schief for their interest and useful remarks.

This research was partially supported by INTAS 96-0770 and the Alexander von Humboldt Foundation.

References

[1] Akivis M.A. and Goldberg V.V., Projective differential geometry of submanifolds, Math. Library, V.49, North-Holland, 1993.

[2] Blaschke W., Vorlesungen über Differentialgeometrie, V.3, Springer-Verlag, Berlin, 1929.

[3] Blaschke W. and Bol G., Geometrie der Gewebe, Springer-Verlag, Berlin, 1938.

[4] Blaschke W., Einführung in die Geometrie der Waben, Birkhäuser-Verlag, Basel, Switzerland, 1955.

[5] Bogdanov L.V., Veselov-Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation, Theor. and Math. Phys., 1987, V.70, 309-314.

[6] Bol G., Projektive Differentialgeometrie, Göttingen (1954).

[7] Cartan E., Sur une classe des surfaces apparantées aux surfaces R et aux surfaces de Jonas, Bull. des Sciences 58 (1944).

[8] Cartan E., Sur la déformation projective des surfaces, Ann. de l'Ecole Normale Sup., (3) 37 (1920) 259-356.

[9] Demoulin A., Sur les surfaces R et les surfaces Ω, C.R. 153 (1911) 590, 705, 797, 927.

[10] Demoulin A., Sur deux transformations des surfaces dont les quadriques de Lie n'ont que deux ou trois points caracteristiques, Bull. de l'Acad. Belgique **19** (1933) 479-502, 579-592, 1352-1363.

[11] Eisenhart L.P., Transformations of surfaces, second ed., 1962.

[12] Ferapontov E.V. and Schief W.K., Surfaces of Demoulin: Differential geometry, Bäcklund transformation and Integrability, *Preprint SFB 288 No. 316*, Berlin (1998).

[13] Ferapontov E.V., Stationary Veselov-Novikov equation and isothermally asymptotic surfaces in projective differential geometry, *Preprint SFB 288 No. 318*, Berlin (1998) (DG/9805011).

[14] Finikov S.P., Projective Differential Geometry, Moscow-Leningrad, 1937.

[15] Finikov S.P., Theory of congruences, Moscow-Leningrad, 1950.

[16] Finikov S.P., W-transformations of Cartan manifolds of special projective type, Mat. Sbornik 43 (85), N2 (1957) 169-186.

[17] Fubini G. and Ĉech E., Geometria Proiettiva Differenziale, Bologna: Zanichelli, 1926.

[18] Gambier B., Surfaces de Jonas et surfaces R, Paris C.R. 204 (1937) 1858-1859.

[19] Geidelman R.M., Multidimensional systems R, Uspekhi Mat. Nauk 12, N3 (75), (1957) 285-290.

[20] Godeaux L., La théorie des surfaces et l'espace réglé (Géométrie projective differentielle), Actualités scientifiques et industrielles, N138, Paris, Hermann (1934).

[21] Godeaux L., Sur les surfaces R, Bull. Acad. Belg. 45 (1959) 829-838.

[22] Hudson R.W.H.T., Kummer's quartic surface, Cambridge Univ. Press 1990 (first published 1905).

[23] Jonas H., Über die konstruktion der W-kongruenzen zu einem gegebenen Brennflächenmantel und über die Transformationen der R-flächen, Jahresber. Deutsch. Math. Ver. 29 (1920) 40-74.

[24] Jonas H., Über eine klasse von flächen die ein gegenstück zu den von Demoulin und Tzitzeica betrachteten R-flächen bilden, Sitzungsber. Berl. Math. Ges. 19 (1921) 18-30.

[25] Jonas H., Zur theorie der Flächen mit Gewindekurven als Asymptotenlinien der einen oder der beiden Scharen, Math. Nachrichten, 15 (1956) 209-239.

[26] Kéraval E., Sur les surfaces dont les lignes asymptotiques appartiennent par leurs tangentes à un complexe linéare, Bull. Soc. Math. de France 39 (1911) 134.

[27] Konopelchenko B.G., Induced surfaces and their integrable dynamics, Studies in Appl. Math., 1996, 9-51.

[28] Konopelchenko B.G., Nets in R^3, their integrable evolutions and the DS hierarchy, Phys. letters A, 1993, V.183, 153-159.

[29] Konopelchenko B.G. and Pinkall U., Integrable deformations of affine surfaces via Nizhnik-Veselov-Novikov equation, *Preprint SFB 288 No. 318*, Berlin (1998).

[30] Konopelchenko B.G. and Rogers C., On generalized Loewner systems: novel integrable equations in 2+1 dimensions, J. Math. Phys., 34 (1993) 214-242.

[31] Korovin V.I., System R in four-dimensional projective space, Doklady AN SSSR, 101, N5 (1955) 797-799.

[32] Kovancov N.I., On a class of surfaces in projective differential geometry, Ukr. Mat. ZH., 4 (1952) 137-154.

[33] Lane E., Treatise on Projective Differential Geometry, The Univ. of Chicago Press (1942).

[34] Lane E., The contact of a cubic surface with an analytic surface, Amer. Math. Soc. Trans., 29 (1927) 471-480.

[35] Lelieuvre M., Sur les lignes asymptotiques et leur représentation sphérique, Bull. Sci. Mathematiques 12 (1888) 126-128.

[36] Lingenberg W., Zur Differentialgeometrie der Flächen, die eine eingliedrige projektive Gruppe in sich gestatten und über allgemeine Projektivrotationsflächen, Math. Z, 66 (1957) 409-446.

[37] Lingenberg W., Über die projektivabvickelbaren isotherm-asymptotischen Flächen, Archiv der Mathematik, 16 (1965) 274-287.

[38] Matveev V.B. and Salle M.A., Darboux transformations and Solitons, Springer Verlag, 1991.

[39] Mayer O., Contribution à l'étude des surfaces minima projectives, Bull. des Sciences Math. ser.2, 56 (1932) 146-168, 188-200.

[40] Mihailescu T., Les surfaces R_0, Czechoslovak Math. J. 8 (83) (1958) 573-582.

[41] The reduction problem and the inverse scattering method, Physica D 3 (1981) 73-117.

[42] Nizhnik L.P., Integration of multidimensional nonlinear equations by the method of inverse problem, DAN SSSR, 254 (1980) 332.

[43] Pa Chenkuo, A transformation of Jonas surfaces, Bull. AMS 49 (1943) 793-796.

[44] Rozet O., Sur certaines congruences W attachée aux surfaces dont les quadriques de Lie n'ont que deux points caracteristiques, Bull. Sci. Math. **II 58** (1934) 141-151.

[45] Sasaki T., On a projectively minimal hypersurface in the unimodular affine space, Geometriae Dedicata 23 (1987) 237-251.

[46] Sasaki T., Projective differential geometry and linear homogeneous differential equations, Rokko lectures in Math., Kobe University, Japan, 5 (1999) 1-115, ISBN 4-907719-05-1.

[47] Sasaki T. and Yoshida M., Linear differential equations in two variables of rank 4, Math. Ann. 282 (1988) 69-93.

[48] Schief W.K., On the geometry of an integrable 2+1-dimensional sine-Gordon system, Proc. Roy. Soc. London A 453 (1997) 1671-1688.

[49] Shulman T.A., Asymptotic transformations of triply-conjugate systems of surfaces. DAN SSSR 85 (1952) 501-504.

[50] Shulman T.A., Triply-conjugate systems of surfaces R. Asymptotic transformations of systems R. Mat. Sbornik 39 (81), (1956) 293-314.

[51] Taimanov I.A., Modified Novikov-Veselov equation and differential geometry of surfaces, in Solitons, Geometry and Topology (eds. V.M. Buchstaber and S.P. Novikov) Transl. AMS, ser.2, 179 (1997) 133-155.

[52] Taimanov I.A., Surfaces of revolution in terms of solitons, Preprint (dg-ga/9610013) to appear in Ann. of Global Analysis and Geometry, 15, N5 (1997).

[53] Thomsen G., Sulle superficie minime proiettive, Ann. di Math. 5 (1928) 169-184.

[54] Thomsen G., Über eine liniengeometrische behandlungsweise der projectiven flächentheoie und die projective geometrie der systeme von flächen zweiter ordnung, Abhandl. Math. Sem. Hamburg, 4 (1926) 232-266.

[55] Tsarev S.P., The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph transform, Math. USSR Izv., 1991, V.37, 397-419.

[56] Tzitzeica G., Sur une nouvelle classe de surfaces, C.R. 150 (1910) 955-956.

[57] Tzitzeica G., Sur certains reseaux conjugues, C.R. 152 (1911) 1077, Sur les reseaux R, C.R. 153 (1911) 1127.

[58] Veselov A.P. and Novikov S.P., Finite-gap two-dimensional potential Schrödinger operators. Explicit formulae and evolution equations, DAN SSSR, 279 (1984) 20.

[59] Wilczynski E.I., Projective-differential geometry of curved surfaces, Trans. AMS 8 (1907) 233-260; 9 (1908) 79-120, 293-315.